Contact reductions, mechanics and duality

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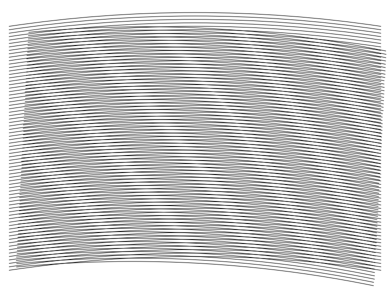
Abstract

Contact reduction is very closely related to symplectic reduction, but it allows symmetries that are not manifest in Hamiltonian mechanics and moreover, solution of the reduced problems yields solution of the original problem without further integration.

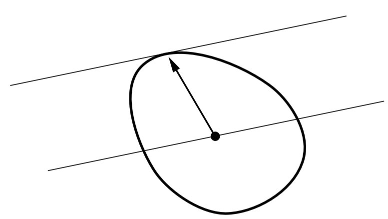
1 An informal introduction: What is the visual meaning of mechanics?

This is a very broad question. We shall be rather modest and content ourselves with the answer found by Hamilton and Jacobi:



According to them, mechanics studies emergence of the thick curves via interference of the thin curves. In higher dimensions, the thin curves are replaced by hypersurfaces (thick curves remain curves). The picture may represent the spacetime with worldlines of particles appearing as wave packets. A bit more abstractly, it may represent an extended configuration space. It may also be the ordinary space with rays of light. And after all, you may see similar phenomena on folded curtains.

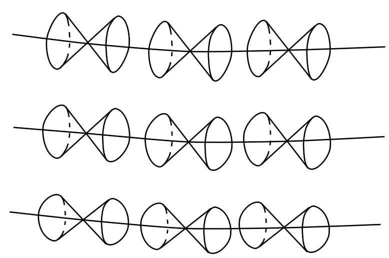
The thin hypersurfaces are supposed to obey certain law (the Hamilton–Jacobi (or eikonal) equation). It may be stated (a bit informally) as follows:



Around each point we have a hypersurface (so-called wave diagram, or indicatrix) and if a thin hypersurface passes through the point, the next one should touch the diagram (the Huyghens principle). The vector represents the direction of the thick curve through the point. Of course, everything is understood infinitesimally: the picture actually represents the tangent space at the point and the hyperplanes are the zero- and the one-level of the differential of the wave phase. The wave diagram

can be used to measure lengths of vectors: by definition, the diagram consists of vectors of the length one. Obviously, the thick curves are extremal with respect to this metric (this is due to the fact that the drawn vector points to the point of tangency). Noether theorem is visually obvious as well: on the first picture you can imagine that we started with one set of thin curves and the other one was obtained by applying an infinitesimal symmetry.

There are several problems that we passed in silence. The most serious one is non-naturalness: in Lagrangian mechanics we may add the differential of any function to the Lagrangian without changing anything essential. It is quite clear what is the corresponding fact in the wave picture: the waves are not just complex functions, but rather sections of a line bundle. Similarly, the phase of a wave is a section of the corresponding principal $U(1)$ bundle. Even if the bundle is trivial, only its bundle structure is natural. This modification may seem minute, but pictures change rather dramatically, as we have one more dimension:



In the principal $U(1)$ bundle P we have an invariant field of cones. To obtain a Lagrangian we choose a local trivialization. Let α be the corresponding flat connection 1-form. We intersect the cones with the levels $\alpha = 1$ and project the intersections to the base; in this way we obtain the wave diagrams.

Hamilton–Jacobi equation says that wave phase (a section of P) should be tangent to the cones. In other words, they are the Monge cones for the H-J equation. The "worldlines" are the (bi)characteristics, i.e. the curves along which the phase touches the cones.

We shall consider this picture as fundamental. We want to find characteristics of a field of cones and it is only a one-parameter group of symmetry that makes it into a variational problem. The space of characteristics is a contact manifold; the $U(1)$ symmetry makes it into a $U(1)$ bundle with a connection (ignoring some obvious problems). From this point of view, this $U(1)$ bundle over the phase space is more fundamental than the phase space itself. Naturality of this approach is advocated further in [3] and it is used in a substantial way in the book [4].

Now imagine that the field of cones is invariant with respect to some (local) Lie group G containing our $U(1)$. This symmetry would be hidden from us if we saw only the base and not the bundle P itself (unless the $U(1)$ were in its centre), even if we passed to Hamiltonian mechanics. We may use G to reduce the problem (via contact reduction). The classical example is for two-dimensional abelian G. We take a vector field v from $\mathfrak{g}$, make quotient of P by v and find a field of cones in the quotient – they are the contours of the original cones when seen in the direction of v. To find the characteristics in P it is sufficient to find them in the reductions and to compute some derivatives. The proper setting for a general G is contact geometry: we want to find the characteristics of a field of hyperplanes and we use a symmetry group G to reduce the problem. Having solved the reductions we solve the original problem simply by computing derivatives. In this respect it is slightly better than symplectic reduction.

And there is one more point: we can take a different $U(1)$ in the group G and make P a bundle in this different way. The two mechanical (or variational)

systems are thus made equivalent. This is the duality mentioned in the title. The two systems have different phase spaces, but they share the $U(1)$-bundle over their phase spaces.

Everything is very simple and we could end here. The remaining sections should only provide examples and more precise definitions and make the paper completely selfcontained. But we should also return to our motivation and explain what is the high-frequency approximation in this picture. Suppose P is a principal $U(1)$ or $\mathbb{R}$ bundle and D is an invariant differential operator $\mathcal{C}^\infty(P) \to \mathcal{C}^\infty(P)$. If we confine ourselves to equivariant functions with some weight $i/\hbar$, D becomes an operator $D_\hbar$ on this associated line bundle. For example, if

$$D = \frac{1}{2m}\Delta + V(x,t)\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial s \partial t}$$

and the group acts by shifting s, $D_\hbar$ is the Schroedinger operator. If $f \in \mathcal{C}^\infty(P)$ satisfies $Df = 0$, each of its Fourier components satisfies $D_\hbar f_\hbar = 0$. So this is the relation classical–quantum in our picture: all admissible $\hbar$'s are collected in a single equation $Df = 0$ and the classical theory describes singularities of solutions of this wave equation.

2 Contact geometry

In this section we review some notions of contact geometry. Fortunately, we need only elementary things. Nice exposition can be found in the EMS article [1].

A contact structure on a manifold M is a maximally non-integrable field of hyperplanes C. For a while, let C denote any subbundle of TM (with arbitrary codimension). If v is a vector field, let $\mathcal{L}_v C : C \to TM/C$ denote the infinitesimal deformation of C by the flow of v. It may be defined by $\mathcal{L}_v C(u) = [v,u] \bmod C$, where u is extended to a section of C in an arbitrary way. If v is a section of C as well, $[v,u] \bmod C$ is completely local in both arguments, so that we have a map $\tau_C : C \wedge C \to TM/C$. If C is a field of hyperplanes (a codimension-one subbundle), it is called maximally nonintegrable if the bilinear form τ_C (with values in the line-bundle TM/C) is everywhere regular. In other words, (M,C) is contact iff for any infinitesimal deformation $C \to TM/C$ there is (unique) section of C with flow generating this deformation.

Stability and characteristics

A simple consequence of the last statement is Gray's stability theorem. If $(M, C(t))$ is a compact manifold with time-dependent contact structure and $C(t)$ is constant on some subset $X \subset M$, there is a flow of M fixing the points of X that generates $C(t)$ from $C(0)$. Compactness is used to ensure completness of vector fields. It is not needed if we want a local statement. For example, if $C(t)$ is fixed at a point (and t restricted to a finite interval), we still have a local diffeomorphism close to the point. As a consequence, all contact structures in a given (necessarily odd) dimension are locally equivalent (Darboux theorem).

And here is another application. A vector field v on a contact manifold (M,C) is *contact*, if its flow preserves C. For any section w of TM/C there is unique contact v equal to $w \bmod C$. To see it, take any v' equal to $w \bmod C$; the deformation $\mathcal{L}_{v'}C$ can be removed by unique section of C. w is called the *contact Hamiltonian* of v.

Contact and symplectic geometries are closely related. We shall need this example. Let $M \to N$ be a pricipal $U(1)$ (or $\mathbb{R}$) bundle and let M be equipped with an

invariant contact structure. Suppose furthemore that the hyperplanes are transversal to the fibres, so that they may be interpreted as a connection. Its curvature is a symplectic form on M. A contact vector field commuting with the $U(1)$ action (i.e. with invariant contact Hamiltonian) is projected to a Hamiltonian vector field on the base: using the connection 1-form to trivialize TM/C, an invariant contact Hamiltonian becomes a function on the base – the sought Hamiltonian. The somewhat mystic generation of vector fields by functions in symplectic geometry is here a bit more visual: Hamiltonian is simply the vertical part of the vector field. Interesting geometry arises even if C is not everywhere transversal to the fibres. The projection of these dangerous points is easily seen to be a smooth hypersurface. The symplectic form diverges at the hypersurface and the hypersurface itself carries a contact structure (this is our first encounter with contact reduction).

Let us now consider a field of hyperplanes C without assuming complete nonintegrability. A section of C lying in the kernel of τ_C is called a *characteristic vector field*. The flow of a characteristic field preserves C. Suppose now that the rank of τ_C is constant. We see that the manifold is locally a product of a contact manifold and $\mathbb{R}^k$ (the characteristic directions), where k is the dimension of the kernel of τ_C. Globally we have a foliation with k-dimensional leaves, called *characteristics*. If the foliation is actually a fibration, its base is contact.

This permits us to see the proof of Gray's theorem and the generation of contact fields by contact Hamiltonians from a "space-time" point of view. Let $(M, C(t))$ be as before. On $M \times \mathbb{R}$ we define a hyperplane field – we simply add $\partial/\partial t$ to $C(t)$. The characteristic curves are the worldlines of the flow. Similarly, for a given section of TM/C on a contact (M, C), we construct a hyperplane field on $M \times \mathbb{R}$ in the obvious way; characterictics are again the worldlines of the flow.

Contact elements and 1st order PDE's

The classical example of a contact manifold is the space of contact elements (i.e. hyperplanes in the tangent space) of a manifold M, which we denote as CM. In other words, CM is the projective bundle associated with T^*M. The field C is given as follows: take an $x \in CM$; it corresponds to a hyperplane H in $T_{\pi(x)}M$, where $\pi : CM \to M$ is the natural projection. Then $(d_x\pi)^{-1}(H)$ is C at x.

Contact geometry, in particular on CM, was invented by Lie to give a geometrical meaning to first order PDE's and to Lagrange method of characteristics. Suppose $E \subset CM$ is a hypersurface; it will represent the equation. Any hypersurface $\Sigma \subset M$ can be lifted to CM: for any point $x \in \Sigma$ take the hyperplane $T_x\Sigma$ to be a point of the lift $\tilde{\Sigma}$. $\tilde{\Sigma}$ is a *Legendre submanifold* of CM, i.e. $T\tilde{\Sigma} \subset C$ and $\tilde{\Sigma}$ has the maximal dimension ($\dim CM = 2\dim\tilde{\Sigma} + 1$). Σ is said to solve the equation if $\tilde{\Sigma} \subset E$. This has a nice interpretation due to Monge: For any $x \in M$ we take the enveloping cone of the hyperplanes $\pi^{-1}(x) \cap E$ in T_xM. In this way we obtain a field of cones in M. Then Σ solves the equation if it is tangent to the cones.

Lie's point of view is to forget about M and to take as a solution any Legendre submanifold contained in E. Such a solution may look singular in M (singularities emerge upon the projection $\pi : CM \to M$; actually, many things classically called functions or hypersurfaces, are in fact Legendre submanifolds of something – we shall meet the example of "generating functions" of contact transformations). This definition uses only the contact structure on CM and thus allows using the entire (pseudo)group of contact transformations.

The hyperplane field C cuts a hyperplane field C_E on E (there may be points where the contact hyperplane touches E; generally they are isolated and we will ignore them). The form τ_C becomes degenerate when restricted from C to C_E; in E, we have one characteristic direction everywhere. For example, if the Monge cones coming from E are the null cones of some pseudo-Riemannian metrics on M then

the projections of the characteristics are the light-like geodesics in M. Legendre submanifolds contained in E are woven from characteristics. To find them (i.e. to solve the equation), we have to take local quotients of E by characteristics and pull their Legendre submanifolds back to E.

Hypersurfaces $E \subset CM$ often come from an equation of the type $Df = 0$, where $D : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is a linear (pseudo)differential operator. We shall be very brief and prove nothing. Take the symbol σ_D of D (a function on T^*M defined by $D \exp(i\lambda g) = (i\lambda)^n \exp(i\lambda g)\sigma_D(dg) + O(\lambda^{n-1})$, $\lambda \to \infty$, where n is the degree of D and $g \in \mathcal{C}^\infty(M)$). The equation $\sigma_D = 0$ specifies a hypersurface $E \subset CM$. Singularities of solutions of $Df = 0$ are located on "hypersurfaces" solving the equation corresponding to E.

Contact reductions and homogeneous spaces

Before proceeding to group actions and reductions, we have to describe the *contact product*. If M_1 and M_2 are contact, there is no contact structure on $M_1 \times M_2$. The contact product $M_1 \times_c M_2$ is actually a circle bundle over $M_1 \times M_2$. In $M_1 \times M_2$ there is an obvious field of codimension-two subspaces. We take all the hyperplanes containing these subspaces – they form the manifold $M_1 \times_c M_2$. Contact structure on $M_1 \times_c M_2$ is defined similarly to the one on CM.

If $\phi : M_1 \to M_2$ is a local diffeomorphism preserving the contact structure then the graph of ϕ in $M_1 \times M_2$ can be uniquely lifted to a Legendre submanifold of $M_1 \times_c M_2$. Vice versa, any Legendre submanifold $L \subset M_1 \times_c M_2$ whose projection is the graph of something gives rise to such a ϕ. L is called (a bit improperly) the *generating function* of ϕ. Notice that $CM \times_c CN = C(M \times N)$; the generating function of a $\phi : CM \to CN$ is therefore a "hypersurface" in $M \times N$.

Like in symplectic geometry, it is natural to call Legendre submanifolds of $M_1 \times_c M_2 \times_c \ldots \times_c M_k$ contact relations; they can be composed under some qualifications on intersections. And similarly to $C(M \times M)$, Legendre submanifolds of $C\Gamma$, where Γ is a Lie groupoid, can be composed. This represents composition of singularities in the groupoid algebra. The reader may like to define contact groupoids.

Now we classify contact homogeneous spaces. An invariant contact structure on G/H can be pulled back to a left-invariand hyperplane field on G; its characteristics are cosets of H. Hence we are done: local G-homogeneous contact spaces are classified by hyperplanes in $\mathfrak{g}$. If $\mathfrak{l} \subset \mathfrak{g}$ is such a subspace, let $\mathfrak{g}_\mathfrak{l}$ be the stabilizer $\mathfrak{S}(\mathfrak{l}) = \{x \in \mathfrak{g};\ [x, \mathfrak{l}] \subset \mathfrak{l}\}$ of $\mathfrak{l}$ intersected with $\mathfrak{l}$; set $H = \exp(\mathfrak{g}_\mathfrak{l})$. For global Lie groups there is no reason why this H should be closed. G-homogeneous contact spaces are classified by pairs $(\mathfrak{l}, H)$, where $H \subset G$ is a closed subgroup with the Lie algebra $\mathfrak{g}_\mathfrak{l}$.

These spaces are closely related to coadjoint orbits. Choose an $\alpha \in \mathfrak{g}^*$ with the kernel $\mathfrak{l}$. If the orbit $\mathcal{O}_\alpha$ is homogeneous then $G/H = \mathcal{O}_\alpha/\mathbb{R}^*$ or $\mathcal{O}_\alpha/\mathbb{R}_+$ (with the obvious contact structure coming from the symplectic form on $\mathcal{O}_\alpha$). If it is not, G/H is a principal bundle with one-parameter group over $\mathcal{O}_\alpha$. These two possibilities appear according to whether $\mathfrak{S}(\mathfrak{l})$ is contained in $\mathfrak{l}$ or not; in the second case, $\mathfrak{S}(\mathfrak{l})/\mathfrak{g}_\mathfrak{l}$ is the algebra of the structure group. The constrains of globality are well visible here: in the first case there are none while in the second case the periods of the symplectic form on $\mathcal{O}_\alpha$ have to be co-mesurable (and it is also sufficient if G is simply-conected).

Now we shall discuss contact reductions. Suppose G acts on a contact space M respecting the contact structure. Let $M_\mathfrak{g}$ consist of the points of M where the image of $\mathfrak{g}$ is contained in C. If the action of G is locally free, $M_\mathfrak{g}$ is a submanifold, C cuts a hyperplane field there and its characteristics are the orbits of G. If G acts properly there (so that $M_\mathfrak{g}/G$ is a manifold), $M//G = M_\mathfrak{g}/G$ is the *contact reduction*; $\dim M//G = \dim M - 2\dim G$. Also notice that G-invariant Legendre

sumbanifolds of M are contained in $M_{\mathfrak{g}}$ and they are the preimages of Legendre submanifolds of $M//G$.

These claims are easy to see. The contact Hamiltonians of $\mathfrak{g}$ vanish at $M_{\mathfrak{g}}$; because the action is locally free, their differentials are linearly independent there. It follows that $M_{\mathfrak{g}}$ is a submanifold of codimension equal to $\dim G$ and it is nowhere touched by C. The vector fields generated by $\mathfrak{g}$ are characteristic on $M_{\mathfrak{g}}$ (they preserve the hyperplane field); for dimensional reasons they generate all the characteristic fields.

To reduce all the parts of M (not just $M_{\mathfrak{g}}$) we are sometimes forced to leave global geometry. Let G be a (local) group acting locally freely on M. For any point $x \in M$ let $\mathfrak{l}(x)$ consist of the vectors in $\mathfrak{g}$ mapped into C at x, and for any hyperplane $\mathfrak{l} \subset \mathfrak{g}$ let $M_{\mathfrak{l}} \subset M$ consist of x's where $\mathfrak{l}(x) = \mathfrak{l}$. Then

$$M//_{\mathfrak{l}}G = M_{\mathfrak{l}}/\exp \mathfrak{g}_{\mathfrak{l}} = (M \times_c (G/\exp \mathfrak{g}_{\mathfrak{l}}))//G$$

is the contact reduction at $\mathfrak{l}$; if $\exp \mathfrak{g}_{\mathfrak{l}}$ is not closed, it makes sense only locally. For example, $CG//_{\mathfrak{l}}G$ is the locally homogeneous space corresponding to $\mathfrak{l}$. If $\mathfrak{S}(\mathfrak{l})$ is not contained in $\mathfrak{l}$, $\mathfrak{S}(\mathfrak{l})/\mathfrak{g}_{\mathfrak{l}}$ is a residual one-parameter symmetry making $M//_{\mathfrak{l}}G$ locally into a principal bundle over a symplectic space.

Let us describe two examples connected with symplectic geometry. Let $M_i \to N_i$ be $U(1)$ (or $\mathbb{R}$) bundles with contact M's and symplectic N's, as above. Let $U(1)$ act diagonally on $M_1 \times_c M_2$; then $(M_1 \times_c M_2)//U(1) = (M_1 \times M_2)/U(1)$. The generating function (recall that it is in fact a Legendre submanifold) of a $U(1)$ equivariant contact map $\phi : M_1 \to M_2$ is contained in $(M_1 \times_c M_2)_{\mathfrak{u}(1)}$ (as it is $U(1)$ invariant); therefore, it is the preimage of a Legendre submanifold $L \subset (M_1 \times_c M_2)//U(1)$. The map ϕ gives rise to a symplectic map ψ between the bases; L is called a generating function of ψ.

Similarly, if $U(1) \times G$ acts on M (the $U(1)$ is the structure group of the bundle), the contact reductions of M give rise to the sympectic reductions of N (a hyperplane in $\mathbb{R} \times \mathfrak{g}$ is the same as an element of $\mathfrak{g}^*$).

Solving characteristic problem via contact reduction

Finally, let us describe how contact reduction can help us in constructing the characteristic foliation of a hyperplane field. It is very simple: the reduction described above can be applied if C is any hyperplane field (not necessarily contact). We find characteristics in the reduced manifold (this is the reduced problem) and we know that original characteristics are contained in their preimages. The problem is solved at this point: these preimages yield characteristics by a very simple procedure, generalizing the method of complete integrals. This is slightly better than symplectic reduction, since there solving reductions does not solve the original problem (actually, using the connection between symplectic and contact reduction we see that it is only necessary to compute an idefinite integral).

Here are the details. Suppose M is a manifold with a hyperplane field C and that a (local) group G acts on M preserving C. $\mathfrak{l}(x)$ is defined as above; it is constant along the characteristics, because the action of G on M gives an action on the local quotients by characteristics (a contact version of Noether theorem). In other words, the subspaces $M_{\mathfrak{l}}$ contain characteristics. We may take $M_{\mathfrak{l}}/\exp \mathfrak{g}_{\mathfrak{l}}$ and find characteristics there; the characteristics of (M, C) are in their preimages. We suppose that these preimages form a foliation.

We are in this situation: we have a foliation of M with leaves containing the characteristics and moreover the tangent spaces of the leaves are contained in C. The following is a very minute generalization of the method of complete integrals. Take an open subset $U \subset M$ where the foliation becomes a fibration $\pi : U \to B$. The hyperplanes C in U are projected to hyperplanes in B. This projected

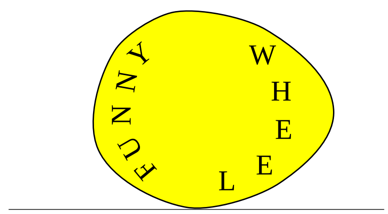
hyperplane is constant along characteristics. And vice versa – this property specifies characteristics. The problem is solved.

3 Examples

We only consider very simple examples: right-invariant hypersurfaces $E \subset CG$ for various Lie groups G. Such a hypersurface is specified by a cone in $\mathfrak{g}^*$ or by its dual (possibly singular) cone in $\mathfrak{g}$; the field of the Monge cones is generated from the latter by right translation. If we choose a (closed, if we want to be global) one-parameter subgroup $R \subset G$, we receive a variational problem on G/R.

To find the characteristics in E we use the contact reduction: we either set $M = E$ in the method described or we make reduction of CG and look what happens with E (these two methods are of course almost identical).

Our first example is the Euclidean group $G = SE(2)$. In this case, there are only two non-conjugate $\mathfrak{l}$'s: either the abelian ideal of $\mathfrak{g}$ or anything else. The first case is trivial (characteristics with this $\mathfrak{l}$ are the straight lines in the abelian normal subgroup (or its cosets) touching the Monge cones). Let us reduce the second case. We find this equipment:



The wheel is funny not just because of its shape. Instead of rolling on the road it remains still and it is the road that rolls around the wheel. Before being more explicit, let us describe the corresponding variational system in the Euclidean plane, i.e. in G/R, where $R = SO(2)$. We draw tangent vectors directly in the plane. To obtain the wave diagram of a point, rotate the funny wheel for 90^{o} around the point. The extremals of this system are the trajectories swept by the points of the plane of the road; among them, the evolvents of the wheel (these are the trajectories of the points lying directly on the road). To find the trajectories it is enough to compute an indefinite integral (the length of arcs of the wheel). The reason behind is the residual one-parameter symmetry mentioned above (we shall make it explicit in a moment).

Now some details. $SE(2)$ is a semidirect product of $SO(2)$ and a vector plane V_2. The algebra $\mathfrak{g}$ is (as a vector space) $\mathbb{R} \oplus V_2$. To find the Monge cone in $\mathfrak{g}$, choose an origin in the plane of the funny wheel (this choice does not matter – we may choose another origin of $SE(2)$), turn it for 90^{o} and place it into the plane $P = \{1\} \times V_2 \subset \mathfrak{g}$; this will be the intersection of P with the Monge cone. The adjoint action of G on P is as the defining affine action, but rotated for 90^{o}. Now we choose an $\mathfrak{l}$; for definiteness, let it contain the algebra of $SO(2)$. We may visualize $G = SE(2)$ with the left-invariant contact structure generated by $\mathfrak{l}$ directly in the plane P; in this way it becomes a double-cover of CP. We represent points of G by flags (point, oriented line though the point). $1 \in G$ is represented by $(\mathfrak{so}(2) \cap P, \mathfrak{l} \cap P)$ (with some orientation of the line); acting on it by a $g \in G$ (adjoint action) we receive the flag of g. For each flag we take the orthogonal contact element – this defines the contact structure. The left action of G is represented by the adjoint action on P and there is a one-parameter right action preserving the contact structure (a line in V_2) – translating the point of the flag along its line.

The contact plane at g touches the Monge cone if the line of its flag touches the wheel. Everything is very simple now. We know the projections of the characteristics to the quotient G/S, where S is the one-parametric right symmetry, i.e. to the space of oriented lines in P (this space inherits a symplectic form from the contact structure on G). The image consists of the lines tangent to the wheel. We have to lift it back to G, moving the point orthogonally to the line. In other words: take an evolvent of the wheel, and make its points into flags by taking the lines normal to the evolvent. This curve in G is a characteristic.

Let us mention that we may choose other R to obtain a variational system – a line in the vector space V_2. In this case, G/R is the space of oriented lines in the Euclidean plane. We have a duality between a variational system on the Euclidean plane and another system on the space of its lines.

The next example is completely trivial. We choose G to be the group of homoteties of an affine space (of arbitrary dimension) and R to be the subgroup fixing a point. G/R is the affine space. The wave diagram is one for all the points (warning: this is *not* translation-invariance). The extremals are simply straight lines. This example is trivial because all the reductions of CG are one-dimensional; there is nothing to compute.

The final example is $G = SO(3)$, $R = SO(2)$. It is very similar to $SE(2)$-case. There is a little problem: to find the Lagrangian (or the wave diagrams) from the Monge cones we need a local trivialization of $G \to G/R$. In the previous examples we used natural global trivializations (the groups were semidirect products). Rather than making arbitrary choices, we use the natural (non-flat) connection (orthogonal to the fibres). We use it to find wave diagrams as we used flat connections. However, its curvature (the area form on S^2) has to be understood as a magnetic field and its potential has to be added to the Lagrangian (so the unnaturality is swept under this rug).

The system looks as follows. Choose a Monge cone in $\mathfrak{g}$. Identify G/R with the unit sphere in $\mathfrak{g}$. To find the wave diagram at a point of the sphere, take the tangent plane, intersect with the cone and rotate for 90°. This time all the $\mathfrak{l}$'s are mutually conjugated. The funny wheel idea works as before (the wheel is the intersection of the sphere with the Monge cone), just instead of planes and lines we have spheres and great circles. The problem is reduced to computing lengths of arcs of the wheel. Characteristics are closed if the perimeter of the wheel is a rational multiple of π.

There is an alternative description of the way the road sphere moves on the wheel sphere. Intersect the dual cone with the sphere and consider the paralel transport along the curve; it extends naturally to a motion of the sphere. This is familiar from the motion of rigid body [2]. It is no accident: the body is described by a field of cones in $SO(3) \times \mathbb{R}^2$ (one $\mathbb{R}$ is the time and the other is the action); the reductions of this group are as those of $SO(3)$.

References

[1] V.I. Arnold, A.B. Givental, *Symplectic geometry*, in Encyclopaedia of Mathematical Sciences 4, Springer 1990

[2] L.D. Landau, E.M. Lifshitz, *Mechanics*, Nauka, Moscow 1973

[3] P. Ševera, *Contact geometry in Lagrangean mechanics*, J.Geom.Phys.(29)3 (1999) pp. 235-242

[4] J.-M.Souriau, *Structure of dynamical systems*, Birkhäuser, Boston 1997

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